IJITCE

ISSN 2347- 3657

# International Journal of Information Technology & Computer Engineering


Email : ijitce.editor@gmail.com or editor@ijitce.com

International Journal of
Information Technology & Computer Engineering

ISSN 2347–3657
Volume 8,Issue 4,Dec 2020

# Horizontal Lifts of Tensor Fields to the Complex Cotangent Bundle

**Dr. B.P. Yadav**

**Abstract**

In this paper, horizontal lifts of tensor fields of a complex manifold $M_{2n}$ to is cotangent bundle $T^2M_{2n}$ are studied.

## 1. Introduction:

Several authors have introduced horizontal lifts of the cotangent bundle $T^2M_{2n}$ of a smooth manifold M using notations of horizontal lifts of tensor field on manifold M, but no natural conjecture has been presented for study of complex structure on cotangent bundle. This demands introduction of some new construction, which we shall prefer to call the construction of complex analytic cotangent bundle of a complex manifold and in brief complex cotangent bundle by $T^2M_{2n}$.

Department of Mathematics
Allahabad Degree College, Allahabad

International Journal of Information Technology & Computer Engineering
ISSN 2347–3657
Volume 8,Issue 4,Dec 2020

2. **Horizontal Lifts of tensor fields on** $T^2 M_{2n}$ :

Let us assume a tensor field, which is pure in its all indices, i.e.

$$S = \left( S_{a_s \ldots\ldots\ldots a_1} dz^{a_s} \otimes \ldots\ldots\ldots\ldots \otimes dz^{a_1} \right)$$

in U of the complex manifold $M_{2n}$ .

we define $S^V$ in $T^2 M_{2n}$ by

(2.1)
$$\gamma S = S^V = \left( v_\mu S_{a_s \ldots\ldots\ldots a_1} \frac{\partial}{\partial v_{a_s}} \otimes V \ldots\ldots\ldots\ldots \otimes \frac{\partial}{\partial v_{a_1}} \right)$$

$$\gamma \overline{S} = S^V = \left( v_\mu S_{\bar{a}_s \ldots\ldots\ldots \bar{a}_1} \frac{\partial}{\partial v_{\bar{a}_s}} \otimes V \ldots\ldots\ldots\ldots \otimes \frac{\partial}{\partial v_{\bar{a}_1}} \right)$$

with respect to the induced coordinates $(z^\alpha, v_\alpha)$, $(z^{\bar{\alpha}}, v_{\bar{\alpha}})$ in $\pi^{-1}(U)$, U being an arbitrary coordinates neighborhood of $M_{2n}$ .The tensor field $\gamma S$ defined in each $\pi^{-1}(U)$ is global tensor in $T^2 M_{2n}$ ,the definition of the $\gamma$ depends the position of covariant index $\mu$ . But we shall apply the operator $\gamma$ exclusively on a tensor field of type (1,1) which has only on covariant index.

Suppose $\nabla$ is affine connection in complex manifold $M_{2n}$, we define a tensor field $S \in T_s^1(M_{2n})$ .If S has components $S_{a_s \ldots\ldots\ldots a_1} \mu, S_{\bar{a}_s \ldots\ldots\ldots \bar{a}_1} \overline{\mu}$ in the neighborhood U of $M_{2n}$, the this tensor field has local expression

$$S = v_\mu S_{a_s \ldots\ldots\ldots a_1} \mu (dz^{a_s}) \otimes \ldots\ldots\ldots \otimes (dz^{a_1})$$

$$\overline{S} = v_{\bar{a}} S_{\bar{a}_s \ldots\ldots\ldots \bar{a}_1} \overline{\mu} (dz^{a_s}) \otimes \ldots\ldots\ldots \otimes (dz^{a_1})$$

with respect to induced coordinate in $\pi^{-1}(U)$ determines a global tensor field in $T^2 M_{2n}$

Other than a function in the complex manifold $M_{2n}$, a tensor field $\nabla \gamma S$ in $T^2 M_{2n}$ is defined by

ISSN 2347–3657
Volume 8,Issue 4,Dec 2020

(2.2) $$\gamma S = S^V = \left(v_\mu \nabla_\gamma S_{a_s \ldots\ldots\ldots\ldots a_1} \mu(dz^{a_s}) \otimes V \ldots\ldots\ldots\ldots \otimes (dz^{a_1})\right)$$

$$\overline{S}^V = v_{\bar{\mu}} \overline{\nabla}_\gamma S_{\bar{a}_s \ldots\ldots\ldots\ldots \bar{a}_1} \overline{\mu}(dz^{a_s}) \otimes \ldots\ldots\ldots \otimes (dz^{a_1}) S_{\bar{a}_s \ldots\ldots\ldots\ldots \bar{a}_1}$$ with respect to the induced coordinates $(z^\alpha, v_\alpha)$, $(z^{\bar{\alpha}}, v_{\bar{\alpha}})$ in $\pi^{-1}(U)$, where $S_{a_s \ldots\ldots\ldots\ldots a_1} \mu$ and $S_{\bar{a}_s \ldots\ldots\ldots\ldots \bar{a}_1} \overline{\mu}$ are components o S in U.

For a function Z in the complex manifold $\mathrm{M}_{2n}$, we put

(2.3) $$(\nabla Z)^C = Z^C$$

from (2.2)and (2.3), we have

(2.4) $$\nabla\gamma(P \otimes Q) = (\nabla\gamma P) \otimes Q^V + P^V \otimes (\nabla\gamma P)$$

for any tensor field P and Q in $\mathrm{M}_{2n}$.

Now we define the horizontal lift $S^H$ of a tensor field S given in $\mathrm{M}_{2n}$ by

(2.5) $$\mathrm{S}^H = S^C + \gamma(\nabla S)$$

Where $\nabla$ is the affine connection in $\mathrm{M}_{2n}$ defined by

$$\nabla_Z V = \nabla_V Z + [Z, V]$$

any $Z, V \in T_0^1(M_{2n})$.Thus the horizontal lifts $\mathrm{S}^H$ coincides with the complete $S^C$ if and only if $\nabla \mathrm{S} = 0$.

Taking account of (2.3) and (2.5) we have

(2.6) $$\mathrm{S}^H = S^C$$

Horizontal lift of tensor field of type (1, 0), i.e. $Z^H$ has components of the form

ISSN 2347–3657
Volume 8,Issue 4,Dec 2020

(2.7) $$Z^H = \begin{bmatrix} z^{\beta} & z^{\bar{\beta}} \\ \Gamma_{\alpha\beta} z^{\gamma} & \Gamma_{\bar{\alpha}\bar{\beta}} z^{\bar{\gamma}} \end{bmatrix}$$

with respect to induced coordinates $((z^{\alpha}, v_{\alpha}), (z^{\bar{\alpha}}, v_{\bar{\alpha}}))$ in $T^2(M_{2n})$, where

(2.8) $\Gamma_{\eta\alpha} = v\mu\Gamma^{\mu}_{\eta\alpha}$ and $\Gamma_{\bar{\eta}\bar{\alpha}} = v_{\bar{\mu}}\Gamma^{\bar{\mu}}_{\bar{\eta}\bar{\alpha}}$

are components of $\nabla$ in $M_{2n}$.

If we take n linearly independent 1-form $\omega_{1,}......, \omega_n,$ in a coordinate neighborhood of U of $M_{2n}$, then their vertical lifts $\omega^V{}_{1,}......, \omega_n{}^V$ are also linearly independent. Next, if we take n linearly independent vector fields $Z_1,....., Z_n$ in U, then their horizontal lifts $Z_1{}^H,....., Z_n{}^H$ are also linearly independent. Moreover, the vertical lift $\omega^V$ of a non zero 1-form $\omega$ with local components $(\omega_{\alpha}, \omega_{\bar{\alpha}})$ has components of the form

(2.9) $$\omega^V = \begin{bmatrix} 0 & 0 \\ \omega_{\alpha} & \omega_{\bar{\alpha}} \end{bmatrix}$$

and the horizontal lift $Z^H$ of a non –zero vector field Z with local coordinates $(z^{\alpha}, z^{\bar{\alpha}})$ has the components of form (1.7). Then $\omega^V$ and $Z^H$ are never linearly dependent. Hence

$\omega^V{}_{1,}......, \omega_n{}^V; Z_1{}^H,....., Z_n{}^H$ are $2n \times 2n$ linearly field independent vector in $\pi^{-1}(U)$ .Thus, we have

**Theorem 2.1**: If S and T be two tensor fields of type (0,s) of type (1,s), where S>0, such that

$$\bar{S}(\overline{Z_s},........., \overline{Z_1}) = \bar{T}(\overline{Z_s},........., \overline{Z_1})$$

for all vector fields $\overline{Z_1},.........\overline{Z_s}$ ,which are of the form $\omega^V$ or $Z^H$, where $\omega \in T_1^0(M_{2n})$ and $Z \in T_0^1(M_{2n})$, then

ISSN 2347–3657
Volume 8,Issue 4,Dec 2020

(2.10) $$\overline{S} = \overline{T}$$

Let F be a tensor field of type (1,1) and $\nabla$ be a symmetric affine connection in complex manifold $M_{2n}$, then we write

(2.11) $$F^H = F^C + \lambda(\nabla \mathrm{F})$$

where $[\nabla \mathrm{F}]$ is a tensor filed of type(1,2) defined by

(2.12) $$(\nabla \mathrm{F})(\mathrm{Z}, \mathrm{V}) = -\nabla_Z (FV) + \nabla_V (FZ)$$

$Z$ and $V$ being arbitrary element of $T_1^0(M_{2n})$. We call $F^H$ the horizontal lift of the tensor field F of type (1,1) in $M_{2n}$ to $T^2 M_{2n}$.The horizontal lift $F^H$ has components of the form

(2.13) $$F^H = \begin{bmatrix} F_\alpha^\beta & F_\alpha^{\overline{\beta}} & 0 & 0 \\ F_{\overline{\alpha}}^\beta & F_{\overline{\alpha}}^{\overline{\beta}} & 0 & 0 \\ \Gamma_{\beta\mu} F_\alpha^\mu - \Gamma_{\alpha\mu} F_\beta^\mu & \Gamma_{\beta\overline{\mu}} F_\alpha^{\overline{\mu}} - \Gamma_{\alpha\overline{\mu}} F_\beta^{\overline{\mu}} & F_\alpha^\beta & F_\alpha^{\overline{\beta}} \\ \Gamma_{\beta\mu} F_{\overline{\alpha}}^\mu - \Gamma_{\overline{\alpha}\mu} F_{\overline{\beta}}^\mu & \Gamma_{\overline{\beta}\overline{\mu}} F_\alpha^{\overline{\mu}} - \Gamma_{\overline{\alpha}\overline{\mu}} F_\beta^\mu & F_{\overline{\alpha}}^\beta & F_{\overline{\alpha}}^{\overline{\beta}} \end{bmatrix}$$

with respect to induced coordinates $(z^\alpha, v^\alpha)$, $(z^{\overline{\alpha}}, v^{\overline{\alpha}})$ in $T^2 M_{2n}$, where $F_\alpha^\beta, F_\alpha^{\overline{\beta}}, F_{\overline{\alpha}}^\beta, F_{\overline{\alpha}}^{\overline{\beta}}$ are local components of F, $\Gamma_{\eta\alpha}^\beta, \Gamma_{\overline{\eta}\overline{\alpha}}^{\overline{\beta}}$ are components of $\nabla$ in $M_{2n}$ and $\Gamma_{\eta\alpha}$,, $\Gamma_{\overline{\eta}\overline{\alpha}}$, are defined in (2.8).

From (2.7),(2.9) and (2.13), we have

**Theorem 2.2**: If $Z \in T_0^1(M_{2n})$, $\omega \in T_1^0(M_{2n})$ and, then $F \in T_1^1(M_{2n})$

(2.14) $$F^H \omega^V = \gamma(\omega \circ F)$$

(2.15) $$F^H Z^H = (FZ)^H$$

(2.16) $$F^H Z^C = (FZ)^H - (\gamma \nabla Z)(F)$$

If $\omega \in T_1^0(M_{2n})$, then by (1.15), we have

$$F^H G^H \omega^V = F^H (\omega \circ F)^V = (\omega \circ GF)^V = (GF)^H \omega^V$$

So that

(2.17) $$(F^H G^H + G^H)\omega^H = (FG + GH)^H \omega^V$$

If $Z \in T_0^1(M_{2n})$, then by (1.16), we have

$$(F^H G^H Z^H) = F^H (GZ)^H = (FGH)^H = (FG)^H \mathrm{Z}^H$$

So that

(2.18) $$(F^H G^H + G^H \mathrm{F}^H)\mathrm{Z}^H = (FG + GF)^H \omega^H$$

As a consequence of (2.17), (2.18) and theorem 1.1, we have

**Theorem 2.3:** if $F, G \in T_1^1(M_{2n})$, then

(2.19) $$F^H G^H + G^H \mathrm{F}^H = (FG + GF)^H$$

**References**

[1] Yano, K. : Differential Geometry on complex and almost complex spaces, Pergamon Press

[2] Yano, K. and S. Kobayashi: Prolongation of tensor fields and connections to tangent bundles, II affine automorphism J. of mathematical .Japan 18 (1996) 236-246 .

[3] Sato I., Almost analytic vector field in almost complex manifold, Tohoku Math. J. 17 (1965), 185-199. [4] Steenrod, N.: The topology of fiber bundles, Princeton Univ., Press, N.J. (1951).

[5] Yano, K. : The theory of Lie derivatives and application, Amsterdam,(1957).

[6] Yano, K. : Tensor fields and connections on cross-section in the cotangent bundle, Tohuku Math. Jour. 19, (1967)32-48.